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Asymptotics and Representation of Cubic Splines*

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1. INTRODUCTION

The local asymptotic behavior of a cubic spline interpolator (with equal bin size) and its derivatives is determined to the first order precisely in the interior as the bin size tends to zero. It is shown that this asymptotic behavior is independent of the boundary conditions usually made use of (in the case of spline interpolation on a finite interval). However, if the local behavior at the boundary or global asymptotic behavior is of interest, the type of boundary conditions assumed can determine what happens. Precise estimates of global asymptotic behavior are also derived. An explicit and simple representation is given for the cubic spline interpolator on the infinite line as well as the doubly cubic spline interpolator on the infinite plane.

2. LOCAL ASYMPTOTICS AWAY FROM THE BOUNDARY FOR A CUBIC SPLINE ON A FINITE INTERVAL

For convenience let us assume that the function f to be interpolated is defined on the interval [0, 1]. The values of the function

$$y_j = f(x_j), \qquad j = 0, 1, ..., n \tag{1}$$

are given at the evenly spaced knots $x_j = jh$, $h = 1/n$. A cubic spline interpolator $s(x)$ is given in terms of the moments

$$M_j = s''(x_j), \qquad j = 0, 1, ..., n \tag{2}$$

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by

$$s(x) = M_{j-1}\frac{(x_j - x)^3}{6h} + M_j\frac{(x - x_{j-1})^3}{6h} + \left(y_{j-1} - \frac{M_{j-1}h^2}{6}\right)\frac{x_j - x}{h} + \left(y_j - \frac{M_j h^2}{6}\right)\frac{x - x_{j-1}}{h},$$

if $x \in [x_{j-1}, x_j]$. The continuity of $s'(x)$ at the knots x_j, $j = 1,\ldots, n-1$, leads to the equations

$$\tfrac{1}{2}M_{j-1} + 2M_j + \tfrac{1}{2}M_{j+1} = d_j, \qquad j = 1,\ldots, n-1, \tag{4}$$

where

$$d_j = 3\,\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}. \tag{5}$$

There are additional restraints due to boundary conditions. In the case of a periodic spline one has

$$\begin{gathered} M_0 = M_n, \qquad y_0 = y_n, \\ \frac{1}{2}M_{n-1} + 2M_0 + \frac{1}{2}M_1 = d_0 = 3\,\frac{y_1 - 2y_0 + y_{n-1}}{h^2}. \end{gathered} \tag{6}$$

The boundary condition $s'(0) = y_0' = f'(0)$, $s'(1) = y_n' = f'(1)$ leads to

$$\begin{aligned} 2M_0 + M_1 &= \frac{6}{h}\left(\frac{y_1 - y_0}{h} - y_0'\right), \\ M_{n-1} + 2M_n &= \frac{6}{h}\left(y_n' - \frac{y_n - y_{n-1}}{h}\right). \end{aligned} \tag{7}$$

The boundary condition $s''(0) = y_0'' = f''(0)$, $s''(1) = y_n'' = f''(1)$ is clearly

$$\begin{aligned} M_0 &= y_0'', \\ M_n &= y_n''. \end{aligned} \tag{8}$$

A variation of (8) is simply $s''(0) = 0$, $s''(1) = 0$

$$M_0 = 0 = M_n. \tag{9}$$

There are a variety of other boundary conditions under which a number of the results derived are still valid. For convenience, however, the discussion will be limited to the boundary conditions (6), (7), (8), and (9). The following theorem describes the error

$$s^{(j)}(x) - f^{(j)}(x), \qquad j = 0, 1, 2, \tag{10}$$

in a manner that is asymptotically precise to the first order as $n \to \infty$ away from the boundary.

THEOREM 1. *Let f be continuously differentiable up to fourth order on the interval* [0, 1]. *Assume that x, $0 < x < 1$, is fixed and that $j = [nx]$ is the greatest integer less than or equal to x with $x_j = j/n$. Then if $u = (x - x_j)/h$*

$$s''(x) - f''(x) = -\frac{f^{(4)}(x)}{2!} hB_2(u) + o(h^2), \tag{11}$$

$$s'(x) - f'(x) = -\frac{f^{(4)}(x)}{3!} h^3 B_3(u) + o(h^3), \tag{12}$$

and

$$s(x) - f(x) = -\frac{f^{(4)}(x)}{4!} h^4 \left\{B_4(u) + \frac{1}{30}\right\} + o(h^4). \tag{13}$$

The $B_k(u)$'s are Bernoulli polynomials. These results are uniformly valid in each interval $[a, b]$ with $0 < a < b < 1$.

Notice that the Bernoulli polynomials as given here are taken with leading coefficient 1 (see [1]) so that

$$B_2(x) = x^2 - x + \tfrac{1}{6},$$
$$B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x,$$
$$B_4(x) = x^4 - 2x^3 + x^2 - \tfrac{1}{30}.$$

The result (12) has already been derived in [4]. We use a similar argument to obtain (11). Now

$$s''(x) = M_j \frac{x_{j+1} - x}{h} + M_{j+1} \frac{x - x_j}{h} \tag{14}$$

for $x \in [x_j, x_{j+1}]$. If M_j is replaced by $f''(x_j)$ in (14), one obtains

$$\begin{aligned} f''(x_j) \frac{x_{j+1} - x}{h} &+ f''(x_{j+1}) \frac{x - x_j}{h} \\ &= \left[f''(x) + (x_j - x) f^{(3)}(x) + \frac{(x_j - x)^2}{2} f^{(4)}(x) + o(h^2)\right] \frac{x_{j+1} - x}{h} \\ &\quad + \left[f''(x) + (x_{j+1} - x) f^{(3)}(x) + \frac{(x_{j+1} - x)^2}{2} f^{(4)}(x) + o(h^2)\right] \frac{x - x_j}{h} \\ &= f''(x) + \frac{f^{(4)}(x)}{2} (x_{j+1} - x)(x - x_j) + o(h^2). \end{aligned}$$

Since

$$M_j - f''(x_j) = -\tfrac{1}{12} f^{(4)}(x)\, h^2 + o(h^2),$$

(see [4]), the error in replacing M_j by $f''(x_j)$ in (14) is

$$-\frac{1}{12} f^{(4)}(x)\, h^2 \left[\frac{x_{j+1} - x}{h} + \frac{x - x_j}{h}\right] + o(h^2) = -\frac{1}{12} f^{(4)}(x)\, h^2 + o(h^2).$$

Thus

$$s_n''(x) - f''(x) = \frac{f^{(4)}(x)}{2}\left[(x_{j+1} - x)(x - x_j) - \frac{h^2}{6}\right] + o(h^2),$$

if $x \in [x_j, x_{j+1}]$. Of course, relation (13) is obtained by going through a parallel argument. However, it can be obtained directly if we assume continuous differentiability up to fifth order. If we rewrite (12) as

$$s'(x) - f'(x) = f^{(4)}(x)\, a(x) + o(h^3),$$

where $x \in [x_j, x_{j+1}]$ then

$$\begin{aligned} s(x) - f(x) &= \int_{x_j}^{x} [s'(y) - f'(y)]\, dy = \int_{x_j}^{x} f^{(4)}(y)\, a(y)\, dy + o(h^4) \\ &= \int_{x_j}^{x} a(y)\, dy\, f^{(4)}(x)\Big|_{x_j}^{x} - \int_{x_j}^{x}\int_{x_j}^{y} a(v)\, dv\, f^{(5)}(y)\, dy + o(h^4) \\ &= \int_{x_j}^{x} a(y)\, dy\, f^{(4)}(x) + o(h^4). \end{aligned}$$

PROPOSITION 1. *Let f be a continuous function on $(-\infty, \infty)$ with $f(x) = O((x)^k)$ as $|x| \to \infty$ for some positive integer k. Then the cubic spline interpolator $s(x)$ of $f(x)$ with*

$$s(x_j) = f(x_j), \qquad j = \ldots, -1, 0, 1, \ldots$$

is given by (3) *with*

$$M_k = \frac{2\alpha}{1 - \alpha^2} \sum_{j=-\infty}^{\infty} (-1)^j \alpha^{|j|} d_{k+j}, \tag{15}$$

where $\alpha = 2 - 3^{1/2}$ and d_j is given by (5).

This follows directly from the fact that

$$\begin{bmatrix} \cdot & \cdot & \cdot & & & & \\ & \cdot & \cdot & \cdot & 0 & & \\ & & \cdot & \cdot & \cdot & & \\ & \frac{1}{2} & 2 & \frac{1}{2} & & & \\ & & \frac{1}{2} & 2 & \frac{1}{2} & & \\ & & & \cdot & \cdot & \cdot & \\ & 0 & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ M_{i-1} \\ M_i \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ d_{i-1} \\ d_i \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

with the matrix on the left a simple Toeplitz matrix which one can easily invert.

The following corollary is a consequence of Theorem 1 and the proposition.

COROLLARY. *Let f be a function continuously differentiable up to fourth order defined on $(-\infty, \infty)$. If $f^{(4)}(x) = O(|x|^k)$ as $|x| \to \infty$ for some integer $k > 0$ then relations* (11), (12), *and* (13) *hold.*

3. Global Measures of Deviation

We shall try to get estimates of the global measures of deviation

$$\int_0^1 [s^{(j)}(x) - f^{(j)}(x)]^2 \, dx, \qquad j = 0, 1, 2 \tag{16}$$

in this section. This will be carried out for the boundary conditions (6), (7), and (8). Basically the argument runs as follows. The estimates (11–13) are uniformly valid away from the boundary points 0 and 1. Precise estimates will not be obtained in the neighborhood of the boundary. However, the estimates for $s^{(j)}(x) - f^{(j)}(x)$ throughout the interval (even in the neighborhood of the boundary) will be shown to be uniformly of the same order of magnitude as the main term in the estimates obtained in the interior. This is enough to show that the estimates obtained in the interior determine the asymptotic behavior of (16) and so the following result is obtained.

THEOREM 2. *Let f be continuously differentiable up to fourth order on* $[0, 1]$. *If the spline interpolator for f with knots at $x_j = jh$ ($h = 1/n$) satisfies boundary* *condition* (6) *in the case of f with period* 1, *or satisfies boundary conditions* (7) *or* (8) *then*

$$\int_0^1 [s''(x) - f''(x)]^2\,dx = \frac{1}{4!}\frac{h^3}{30}\int_0^1 [f^{(4)}(x)]^2\,dx + o(h^3), \tag{17}$$

$$\int_0^1 [s'(x) - f'(x)]^2\,dx = \frac{1}{6!}\frac{h^5}{42}\int_0^1 [f^{(4)}(x)]^2\,dx + o(h^5), \tag{18}$$

$$\int_0^1 [s(x) - f(x)]^2\,dx = \frac{1}{30}\left\{\frac{(4!)^2}{8!} + \frac{1}{10}\right\}\frac{h^7}{(4!)^2}\int_0^1 [f^{(4)}(x)]^2\,dx + o(h^7) \tag{19}$$

as $h \to 0$.

Let us now indicate how one gets the required bounds for $s^{(j)}(x) - f^{(j)}(x)$. In such an argument one initially replaces the M_j's by $f''(x_j)$ in the expression for $s^{(j)}(x)$ and compares the result with $f^{(j)}(x)$. The argument here is the same as that in the proof of Theorem 1 and one obtains the same estimate. Then one assesses the error made in replacing the M_j's by $f''(x_j)$. It is this part of the argument that yields a different estimate from the corresponding argument for Theorem 1. Equations (4) with the appropriate boundary conditions can be written in matrix form as

$$AM = d, \qquad M = (M_j), \qquad d = (d_j) \tag{20}$$

with

$$r = d - AG = A(M - G), \tag{21}$$

where

$$G = (f''(x_j)). \tag{22}$$

With the boundary conditions (6), (7), and (8) it can be shown that $r_j = O(h^2)$ uniformly in j. Further in

$$M_j - f''(x_j) = \sum_k A^{-1}_{j,k} r_k \tag{23}$$

one has uniform absolutely summability of $\sum_k A^{-1}_{j,k}$ in j and n (see [2] for explicit forms for A^{-1}). This implies that $s^{(j)}(x) - f^{(j)}(x) = O(h^{4-j}), j = 0, 1, 2$ uniformly in x. A convenient result obtained from [1] is that

$$\int_0^1 B_n(t)\,B_m(t)\,dt = (-1)^{n-1}\frac{m!\,n!}{(m+n)!}\,B_{m+n}\,.$$

Since

$$B_4 = -\frac{1}{30} = B_8\,, \qquad B_6 = \frac{1}{42}\,,$$

we have

$$\int_0^1 B_2(x)^2\,dx = \frac{(2!)^2}{4!}\frac{1}{30},$$

$$\int_0^1 B_3(x)^2\,dx = \frac{(3!)^2}{6!}\frac{1}{42},$$

$$\int_0^1 \left[B_4(x) + \frac{1}{30}\right]\,dx = \frac{1}{30}\left\{\frac{(4!)^2}{8!} + \frac{1}{10}\right\}.$$

4. More on Boundary Behavior

In some situations $f(x_j)$, $j = 0, 1,..., n$, is given but f is not periodic and there is no a priori information on the derivatives of f at the boundary points 0 and 1. Boundary condition (9) (see [3]) has been proposed because it arises naturally for the spline $s(x)$ interpolating f with knots at the x_j's on [0, 1] that minimizes

$$\int_0^1 [s'(x)]^2\,dx.$$

It is of some interest to examine the local behavior of this spline in the neighborhood of the boundaries 0 and 1 as well as the global measures of deviation (16) and contrast it with what has already been obtained for the boundary conditions (6), (7), and (8).

Theorem 3. *Let f be continuously differentiable up to fourth order on* [0, 1]. *For the spline interpolator $s(x)$ of $f(x)$ with knots x_j, $j = 0, 1,..., n$, and boundary condition* (9) *we then have*

$$M_j - f''(x_j) = -\sigma^j f''(0) - \sigma^{n-j} f''(1) + O(h^2), \tag{24}$$

as $n \to \infty$, where $\sigma = -\alpha$ and the O is uniform in j. This implies that

$$s^{(k)}(x) - f^{(k)}(x) = \sigma^j f''(0)\,A^{(k)}(x) + \sigma^{n-j} f''(1)\,B_j^{(k)}(x) + O(h^{4-k}), \tag{25}$$
$$k = 0, 1, 2 \text{ for } x \in [x_j, x_{j+1}], j = 0, 1,..., n-1,$$

with O uniform in j, and

$$\begin{aligned} A_j(x) &= \left\{\sigma\frac{(x - x_j)^3}{6h} - \frac{(x_{j+1} - x)^3}{6h} + \frac{h}{6}(x_{j+1} - x + \sigma[x - x_j])\right\}, \\ B_j(x) &= \left\{\sigma^{-1}\frac{(x - x_j)^3}{6h} - \frac{(x_{j+1} - x)^3}{6h} + \frac{h}{6}(x_{j+1} - x + \sigma^{-1}[x - x_{j-1}])\right\}. \end{aligned} \tag{26}$$

Further

$$\int_0^1 (s^{(j)}(x) - f^{(j)}(x))^2\, dx = c_j[f''(0)^2 + f''(1)^2]\, h^{5-2j} + O(h^{7-2j}), \quad (27)$$

as $h \to 0$ *with* $c_j > 0$ *a constant*, $j = 0, 1, 2$.

The estimate in formula (24) follows almost immediately from (20), (21), and (22) and the explicit evaluation of A^{-1} in this case (see [2, p. 39]). One then simply has to insert (24) in the expression for $s^{(k)}(x)$ to get the estimate (25).

It is clear from Theorem 3 that for the cubic spline interpolator $s(x)$ with boundary conditions (9), the local deviations (25) in the neighborhood of 0 or 1 as well as the global deviations (16) are of larger order of magnitude than they are for the spline interpolator with boundary conditions (6), (7), or (8). However, one plausible approach is, for example, to consider a spline with a boundary condition of the form (7) where y_0', y_n' are replaced by linear forms y_0^*, y_n^* in the available data $f(x_j)$, $j = 0, 1, \ldots, n$, that are very good estimates of $f'(0)$, $f'(1)$ (which are not available). Specifically we propose

$$\begin{aligned} y_0^* &= 3h^{-1}[f(h) - f(0)] - \frac{3}{2}h^{-1}[f(2h) - f(0)] \\ &\quad + \frac{h^{-1}}{3}[f(3h) - f(0)], \\ y_n^* &= 3h^{-1}[f(1) - f(1-h)] - \frac{3}{2}h^{-1}[f(1) - f(1-2h)] \\ &\quad + \frac{h^{-1}}{3}[f(1) - f(1-3h)]. \end{aligned} \quad (28)$$

The following proposition can be obtained.

PROPOSITION 2. *Let* $f \in C^4[0, 1]$. *Assume that* $f(x_j)$, $j = 0, 1, \ldots, n$ *are given. Consider the cubic spline interpolator* $\bar{s}(x)$ *of* $f(x)$ *satisfying the boundary conditions*

$$\begin{aligned} 2M_0 + M_1 &= \frac{6}{h}\left(\frac{y_1 - y_0}{h} - y_0^*\right), \\ M_{n-1} + 2M_n &= \frac{6}{h}\left(y_n^* - \frac{y_n - y_{n-1}}{h}\right). \end{aligned} \quad (29)$$

The conclusions of Theorems 1 *and* 2 *are then valid for* $\bar{s}(x)$.

A simple expression shows that

$$y_0' - y_0^* = -\frac{h^3}{4} f^{(4)}(0) + o(h^3)$$

with a corresponding result holding for $y_n' - y_n^*$.

The proposition follows then almost immediately.

5. A Doubly Cubic Spline

In this section an explicit representation will be given for a doubly cubic spline interpolator with the knots on an equally spaced lattice on the whole plane. The interpolation problem for the whole plane has been considered rather than one for a square (or rectangle) so as to avoid the questions concerning boundary conditions at the edges of the square which would in turn lead to a representation (obtainable by the same techniques) that is of a more complicated form. An asymptotic question of the type dealt with in Theorem 1 for the univariate spline can be considered for the bivariate spline.

Let $f(x, y)$ be a continuous function defined on the plane $-\infty < x, y < \infty$. The values of the function f

$$z_{i,j} = f(x_i, y_j), \qquad i, j = ..., -1, 0, 1,... \tag{30}$$

are given on the lattice (x_i, y_j) where $x_i = ih$, $y_j = jh$, $h > 0$. Let

$$R_{i,j} = \{(x, y): x_i \leqslant x \leqslant x_{i+1}, y_j \leqslant y \leqslant x_{j+1}\}. \tag{31}$$

A doubly cubic spline with respect to the lattice (x_i, y_j) is a function that is a double cubic in each rectangle $R_{i,j}$ and that is globally an element of C_2^4 (C_r^n is the collection of functions whose nth order partial derivatives involving no more than rth order differentiation with respect to a single variable, exist and are continuous). Our interest is in a doubly cubic spline interpolator $s(x, y)$ of $f(x, y)$ given the values of f on the lattice, that is,

$$s(x_i, y_j) = f(x_i, y_j) = z_{i,j}.$$

We shall now recall an earlier formula (since it will occur again in the following discussion) and rewrite it in a slightly different form. Formula (15) will be rewritten as

$$M_k(y.) = \frac{2\alpha}{1 - \alpha^2} \sum_{j=-\infty}^{\infty} (-1)^j \alpha^{|j|} d_{k+j}(y.) \tag{32}$$

to emphasize that the M_k's are derived from the sequence y. A two-dimensional analog of (32) computed from the double indexed z data is given by

$$M_{i,j}(z) = \left(\frac{2\alpha}{1-\alpha^2}\right)^2 \sum_{a,b} (-1)^{a+b} \, \alpha^{|a|+|b|} \, d_{i+a} d_{j+b}(z_{\cdot,\cdot}) \tag{33}$$

Here $d_i d_j(z_{\cdot,\cdot})$ just amounts to an iteration of the second difference (see (5)) d operator applied to the second subscript of z at j and then a second difference d operator applied to the first subscript of z at i. It will also be convenient to introduce the cubic polynomial

$$W_i(x) = \frac{(x_i - x)^3}{6h} - \frac{h(x_i - x)}{6}. \tag{34}$$

Enough notation has been introduced to state the following theorem.

THEOREM 4. *Let $f(x, y)$ be a continuous function defined on the plane with $f(x, y) = O(|x|^k + |y|^k)$ as $|x|$ or $|y| \to \infty$ with k a positive integer. If $s(x, y)$ is the doubly cubic spline interpolator of f taking on the values* (30) *at the points (x_i, y_j), it can be written as*

$$\begin{aligned}
s(x, y) = {} & M_{i-1,j-1}(z)\, W_i(x)\, W_j(y) \\
& - M_{i-1,j}(z)\, W_i(x)\, W_{j-1}(y) - M_{i,j-1}(z)\, W_{i-1}(x)\, W_j(y) \\
& + M_{i,j}(z)\, W_{i-1}(x)\, W_{j-1}(y) \\
& + M_{i-1}(z_{\cdot,j-1})\, W_i(x) \frac{y_j - y}{h} + M_{i-1}(z_{\cdot,j})\, W_i(x) \frac{y - y_{j-1}}{h} \\
& - M_i(z_{\cdot,j})\, W_{i-1}(x) \frac{y - y_{j-1}}{h} - M_i(z_{\cdot,j-1})\, W_{i-1}(x) \frac{y_j - y}{h} \\
& + M_{j-1}(z_{i-1,\cdot})\, W_j(y) \frac{x_i - x}{h} - M_j(z_{i-1,\cdot})\, W_{j-1}(y) \frac{x_i - x}{h} \\
& + M_{j-1}(z_{i,\cdot})\, W_j(y) \frac{x - x_{i-1}}{h} - M_j(z_{i,\cdot})\, W_{j-1}(y) \frac{x - x_{i-1}}{h} \\
& + z_{i-1,j-1} \frac{y_j - y}{h} \frac{x_i - x}{h} + z_{i-1,j} \frac{y - y_{j-1}}{h} \frac{x_i - x}{h} \\
& + z_{i,j-1} \frac{y_j - y}{h} \frac{x - x_{i-1}}{h} + z_{i,j} \frac{y - y_{j-1}}{h} \frac{x - x_{i-1}}{h}
\end{aligned}$$

if $x \in [x_{i-1}, x_i]$, $y \in [y_{j-1}, y_j]$.

In obtaining the above representation of $s(x, y)$, one can make use of a useful remark made in [2, Sect. 7.3]. First consider the data $z_{i,j}$ with i fixed and construct a univariate cubic spline $s_i(y)$ that interpolates this data. Then, with y fixed interpolate the values $s_i(y)$ at the points x_i by a cubic spline interpolator $s(x, y)$. This last function regarded as a function of two variables (x, y) is the desired doubly cubic spline. Proposition 1 ensures that this construction is well defined under the conditions $f(x, y) = O(|x|^k + |y|^k)$ as $|x| + |y| \to \infty$ for some $k > 0$.

It is clear that

$$s_i(y) = M_{j-1}(z_{i,\cdot}) \frac{(y_j - y)^3}{6h} + M_j(z_{i,\cdot}) \frac{(y - y_{j-1})^3}{6h} + \left(z_{i,j-1} - \frac{M_{j-1}(z_{i,\cdot})\, h^2}{6}\right) \frac{y_j - y}{h} + \left(z_{i,j} - \frac{M_j(z_{i,\cdot})\, h^2}{6}\right) \frac{y - y_{j-1}}{h} \tag{36}$$

for $y \in [y_{j-1}, y_j]$. We now interpolate the values $s_i(y)$ at x_i (y fixed) by the cubic interpolator $s(x, y)$ and obtain

$$s(x, y) = M_{i-1}(s.(y)) \frac{(x_i - x)^3}{6h} + M_i(s.(y)) \frac{(x - x_{i-1})^3}{6h} + \left(s_{i-1}(y) - \frac{M_{i-1}(s.(y))\, h^2}{6}\right) \frac{x_i - x}{h} + \left(s_i(y) - \frac{M_i(s.(y))\, h^2}{6}\right) \frac{x - x_{i-1}}{h} \tag{37}$$

if $x \in [x_{i-1}, x_i]$. If one inserts the representation

$$M_i(s.(y)) = \frac{2\alpha}{1 - \alpha^2} \sum_{a=-\infty}^{\infty} (-1)^a \alpha^{|a|} d_{i+a}(s.(y)),$$

into (37), with $s.(y)$ as given by (36) and rearranges and consolidates terms appropriately, the expression (35) for $s(x, y)$ is obtained. The derivation of Theorem 4 is complete.

Notice that

$$\begin{aligned} \frac{\partial}{\partial y} \frac{\partial}{\partial x} s(x, y) = {} & M_{i-1,j-1}(z)\, V_i(x)\, V_j(y) - M_{i-1,j}(z)\, V_i(x)\, V_{j-1}(y) \\ & - M_{i,j-1}(z)\, V_{i-1}(x)\, V_j(y) + M_{i,j}(z)\, V_{i-1}(x)\, V_{j-1}(y) \\ & + \frac{1}{h}\{-M_{i-1}(z_{\cdot,j-1})\, V_i(x) + M_{i-1}(z_{\cdot,j})\, V_i(x) \\ & - M_i(z_{\cdot,j})\, V_{i-1}(x) + M_i(z_{\cdot,j-1})\, V_{i-1}(x) \\ & - M_{j-1}(z_{i-1,\cdot})\, V_j(y) + M_j(z_{i-1,\cdot})\, V_{j-1}(y) \\ & + M_{j-1}(z_{i,\cdot})\, V_j(y) - M_j(z_{i,\cdot})\, V_{j-1}(y)\} \\ & + \frac{1}{h^2}(z_{i,j} - z_{i-1,j} - z_{i,j-1} + z_{i-1,j-1}), \end{aligned} \tag{38}$$

where

$$V_i(x) = \frac{\partial}{\partial x} W_i(x) = -\frac{(x_i - x)^2}{2h} + \frac{h}{6},$$

if $x \in [x_{i-1}, x_i]$, $y \in [y_{j-1}, y_j]$.

Let $f(x, y)$ be an element of C_β^β on the plane with the partial derivatives

$$\frac{\partial^a}{\partial x^a} \frac{\partial^{8-a}}{\partial y^{8-a}} f(x, y) = O(|x|^k + |y|^k)$$

as $|x| + |y| \to \infty$ for some $k > 0$, $a = 0, 1, \ldots, 8$. Let $s(x, y)$ be the doubly cubic spline interpolator of $f(x, y)$ taking on the values (30) at the points (x_i, y_j). Then one would expect

$$\frac{\partial}{\partial y} \frac{\partial}{\partial x} (s(x, y) - f(x, y)) = O(h^6) \tag{39}$$

as $h \to 0$. The precise asymptotic form of (39) could be obtained as in the one-dimensional case in the corollary, given some lengthy computations.

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